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Nonresonance on the boundary and strong solutions of elliptic equations with nonlinear boundary conditions

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Abstract

We deal with the solvability of linear second order elliptic partial differential equations with nonlinear boundary conditions by imposing asymptotic nonresonance conditions of nonuniform type with respect to the Steklov spectrum on the boundary nonlinearity. Unlike some recent approaches in the literature for problems with nonlinear boundary conditions, we cast the problem in terms of nonlinear compact perturbations of the identity on appropriate *trace spaces* in order to prove the existence of strong solutions. The proofs are based on *a priori* estimates for possible solutions to a homotopy on suitable trace spaces and topological degree arguments.

1 Introduction

This paper is concerned with existence results for strong solutions of second order elliptic partial differential equations with nonlinear boundary conditions of the form

$$\begin{aligned} -\Delta u + c(x)u &= 0 \quad \text{a.e. in } \Omega, \\ \frac{\partial u}{\partial \nu} &= g(x,u) \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with boundary $\partial\Omega$ of class C^2, $\partial/\partial\nu := \nu \cdot \nabla$ is the outward (unit) normal derivative on $\partial\Omega$, $c \in L^p(\Omega)$, $p > N$, where $c(x) \geq 0$ a.e. in Ω with strict inequality on a subset of Ω of positive measure, and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function with at most linear growth (see below). The case where $c \equiv 0$ (the original Steklov problem concerning harmonic functions) will also be considered; the reader is referred to Remarks 3 and 4 at the end of the paper.
As aforementioned, throughout this paper the boundary nonlinearity $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is assumed to satisfy the following two conditions.

For every constant $r > 0$, there is a constant $K = K(r) > 0$ such that

$$|g(x,u) - g(y,v)| \leq K\left(|x-y| + |u-v|\right) \tag{1.2}$$

for all $x, y \in \partial\Omega$ and all $u, v \in \mathbb{R}$ with $u, v \in [-r, r]$.
There are constants $a, b > 0$ such that

$$|g(x,u)| \leq a + b|u| \tag{1.3}$$

for all $(x,u) \in \partial\Omega \times \mathbb{R}$.
By a (strong) solution to Eq.(1.1) we mean a function $u \in W_p^2(\Omega)$ which satisfies (1.1) (the second equality in (1.1) being satisfied in the sense of trace). The reader is referred for instance to [2, 3, 4, 5, 6, 14] for the definitions and properties of Sobolev trace-spaces used in this paper.
We are mainly interested in the case when the boundary nonlinearity g interacts in some sense with two consecutive eigenvalues of the linear problem

$$\begin{aligned} -\Delta u + c(x)u &= 0 \quad \text{a.e. in } \Omega, \\ \frac{\partial u}{\partial \nu} &= \mu u \quad \text{on } \partial\Omega, \end{aligned} \tag{1.4}$$

where $\mu \in \mathbb{R}$ is a spectral parameter on the boundary which was first introduced on a disk in [15] and, more recently, significantly extended in [1, 9]. More specifically, we consider the case when the nonlinear ratio $g(x,u)/u$ asymptotically stays between two consecutive eigenvalues, but need not be uniformly bounded away from these eigenvalues as was previously required in the literature (see e.g. [1] for the linear case and [10, 11] for both the linear and nonlinear problems, and references therein). To the best of our knowledge this appears to be the first time this sort of conditions are considered between any consecutive Steklov eigenvalues when one has a trace-nonlinearity, unlike the case when one has a reaction nonlinearity in the differential equation (see e.g. [13] and references therein). It should be pointed out that, in this case, we work in a completely different setting since trace-spaces are considered in order to obtain the required *a priori* estimates for the nonlinear problem.
Unlike some recent approaches in the literature for problems with nonlinear boundary conditions, we cast the problem in terms of nonlinear compact perturbations of the identity on appropriate *trace spaces* in order to prove the existence of strong solutions. In Section 2 below, we state the main result and introduce the nonlinear functional analytic setting. The proofs are based on *a priori* estimates (derived herein) for possible solutions to a homotopy on suitable trace spaces and topological degree arguments. Remarks are given at the end of the paper to shed more light on the main result and discuss its variants.

2 Nonuniform Nonresonance

In this section we impose conditions on the asymptotic behavior of the 'slopes' of the boundary nonlinearity $g(x,u)$, i.e., on $g(x,u)/u$ as $|u| \to \infty$. These conditions are of nonuniform type since the asymptotic ratio $g(x,u)/u$ need not be (uniformly) bounded away from consecutive Steklov eigenvalues. We mention that in all the results below, the boundary nonlinearity $g(x,u)$ may be replaced by $g(x,u) + h(x)$ where $h \in W_p^{1-1/p}(\partial\Omega) \subset C(\partial\Omega)$.

In order to prove our results, we take a different approach which is based on topological degree theory on suitable boundary-trace spaces. This is in contrast with some recent approaches where variational methods were used for problems with nonlinear boundary conditions. The main result of this paper is given in the following existence theorem. (The case $c \equiv 0$ will be discussed at the end of the paper; see Remarks 3 and 4.)

Theorem 1 (Nonuniform nonresonance between consecutive Steklov eigenvalues) *Assume there are functions $\alpha, \beta \in L^\infty(\partial\Omega)$ such that*

$$\mu_j \leq \alpha(x) \leq \liminf_{|u|\to\infty} \frac{g(x,u)}{u} \leq \limsup_{|u|\to\infty} \frac{g(x,u)}{u} \leq \beta(x) \leq \mu_{j+1}$$

uniformly for a.e. $x \in \partial\Omega$ with

$$\oint (\alpha(x) - \mu_j)\varphi^2 > 0 \;\forall \varphi \in E_j \setminus \{0\} \quad \textit{and} \quad \oint (\mu_{j+1} - \beta(x))\psi^2 > 0 \;\forall \psi \in E_{j+1} \setminus \{0\},$$

where, for $i \in \mathbb{N}$, E_i denotes the (finite-dimensional) Steklov nullspace associated with the Steklov eigenvalue μ_i. Then, the nonlinear equation (1.1) *has at least one (strong) solution $u \in W_p^2(\Omega)$.*

In contrast to some recent approaches in the literature for problems with nonlinear boundary conditions, we first cast the problem in terms of nonlinear compact perturbations of the identity on appropriate *boundary-trace spaces* as follows.
Set $\sigma := (\mu_j + \mu_{j+1})/2$, we consider the homotopy

$$\begin{aligned} -\Delta u + c(x)u = 0 \quad &\text{a.e. in } \Omega, \\ \frac{\partial u}{\partial \nu} - \sigma u = \lambda[-\sigma u + g(x,u)] \quad &\text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $\lambda \in [0,1]$; or equivalently,

$$\begin{aligned} -\Delta u + c(x)u = 0 \quad &\text{a.e. in } \Omega, \\ \frac{\partial u}{\partial \nu} = (1-\lambda)\sigma u + \lambda g(x,u) \quad &\text{on } \partial\Omega. \end{aligned} \tag{2.2}$$

Note that for $\lambda = 0$ we have a linear problem which admits only the trivial solution since σ is in the resolvent of the linear Steklov problem (see e.g. [1, 9]). Whereas, for $\lambda = 1$, we have Eq.(1.1).
We define the linear (Steklov) boundary operator

$$\mathcal{B} : \ \mathrm{Dom}(\mathcal{B}) \subset W_p^2(\Omega) \Subset W_p^{1-1/p}(\partial\Omega) \to W_p^{1-1/p}(\partial\Omega) \quad \text{by}$$

$$\mathcal{B}u := \frac{\partial u}{\partial \nu} - \sigma u,$$

where

$$\mathrm{Dom}(\mathcal{B}) := \{u \in W_p^2(\Omega)) : -\Delta u + c(x)u = 0 \text{ a.e. in } \Omega\}.$$

Here, the compact 'containment' $W_p^2(\Omega) \Subset W_p^{1-1/p}(\partial\Omega)$ must be understood in the sense of trace; i.e., the trace operator $W_p^2(\Omega) \hookrightarrow W_p^{1-1/p}(\partial\Omega)$ is a compact linear operator (see e.g. [5]).
We now define the nonlinear (Nemytskĭi) operator

$$\mathcal{N} : W_p^{1-1/p}(\partial\Omega) \subset C(\partial\Omega) \to W_p^{1-1/p}(\partial\Omega)$$

by

$$\mathcal{N}u = -\sigma u + g(\cdot, u).$$

Eq.(1.1) is then equivalent to finding $u \in \mathrm{Dom}(\mathcal{B})$ such that

$$\mathcal{B}u = \mathcal{N}u. \tag{2.3}$$

Whereas the homotopy Eq.(2.1) is equivalent to

$$\mathcal{B}u = \lambda\mathcal{N}u, \quad \lambda \in [0,1], \quad u \in \mathrm{Dom}(\mathcal{B}). \tag{2.4}$$

From the above definitions, we deduce the following properties for the linear operator $\mathcal{B}$ and the nonlinear operator $\mathcal{N}$. Observe first that $\mathrm{Dom}(\mathcal{B}) := \{u \in W_p^2(\Omega)) : -\Delta u + c(x)u = 0$ a.e. in $\Omega\}$ is a closed linear subspace of $W_p^2(\Omega)$, and that the linear operator $\mathcal{B} :\ \mathrm{Dom}(\mathcal{B}) \to W_p^{1-1/p}(\partial\Omega)$ is continuous, one-to-one and onto. Thus, it is a Fredholm operator of index zero since the nullspace $\mathrm{Ker}(\mathcal{B}) = \{0\}$ and the range $R(\mathcal{B}) = W_p^{1-1/p}(\partial\Omega)$. Owing to the compactness of the trace operator $\mathrm{Dom}(\mathcal{B}) \hookrightarrow W_p^{1-1/p}(\partial\Omega)$, we deduce that

$$\mathcal{K} := \mathcal{B}^{-1} : W_p^{1-1/p}(\partial\Omega) \to \mathrm{Dom}(\mathcal{B}) \hookrightarrow W_p^{1-1/p}(\partial\Omega)$$

is a compact linear operator from $W_p^{1-1/p}(\partial\Omega)$ into $W_p^{1-1/p}(\partial\Omega)$.
Since the function g is locally Lipschitz and $W_p^{1-1/p}(\partial\Omega) \subset C(\partial\Omega)$ (through the surjectivity of the trace operator $W_p^1(\Omega) \to W_p^{1-1/p}(\partial\Omega)$ and the imbedding $W_p^1(\Omega) \subset\subset C(\overline{\Omega})$ for $p > n$), it follows that the nonlinear operator $\mathcal{N} : W_p^{1-1/p}(\partial\Omega) \to W_p^{1-1/p}(\partial\Omega)$ is continuous, and therefore $\mathcal{K}\mathcal{N} : W_p^{1-1/p}(\partial\Omega) \to W_p^{1-1/p}(\partial\Omega)$ is a nonlinear compact (i.e., completely continuous) operator. Thus, Eq.(2.4) is equivalent to

$$u = \lambda\mathcal{K}\mathcal{N}u, \quad \text{with } \lambda \in [0,1] \text{ and } u \in W_p^{1-1/p}(\partial\Omega); \tag{2.5}$$

which shows that, for each $\lambda \in [0,1]$, the operator $\lambda\mathcal{K}\mathcal{N}$ is a nonlinear compact perturbation of the identity on $W_p^{1-1/p}(\partial\Omega)$. It suffices to show that $\mathcal{K}\mathcal{N}$ has a fixed point u in $W_p^{1-1/p}(\partial\Omega)$. (Notice that, by the properties of $\mathcal{K}$, it follows that such a fixed point u belongs necessarily to $\mathrm{Dom}(\mathcal{B})$. Hence, $u \in W_p^2(\Omega)$ and is a (strong) solution of the nonlinear equation (1.1).)
For this purpose, we show that all possible solutions to the homotopy (2.1) (equivalently, (2.2) and (2.4)) are uniformly bounded in $W_p^{1-1/p}(\partial\Omega)$ independently of $\lambda \in [0,1]$ (actually we show that they are bounded in $W_p^2(\Omega)$ also), and then use topological degree theory to show existence of a strong solution. We first prove the following lemma which provides intermediate *a priori* estimates.

Lemma 1 *Assume that the conditions in Theorem 1 are met. Then all possible solutions to the homotopy (2.2) are (uniformly) bounded in $H^1(\Omega)$ independently of $\lambda \in [0,1]$.*

Proof. Suppose the conclusion of the lemma does not hold. Then, there are sequences $\{u_n\} \subset H^1(\Omega)$ and $\{\lambda_n\} \subset [0,1]$ such that $||u_n||_c \to \infty$ and

$$\int \nabla u_n \nabla v + \int c(x) u_n v = \oint (1-\lambda_n)\sigma u_n v + \oint \lambda_n g(x,u_n) v \quad \text{for all } v \in H^1(\Omega). \tag{2.6}$$

Set $v_n = \dfrac{u_n}{||u_n||_c}$. One sees that v_n is bounded in $H^1(\Omega)$. Therefore, there exists a subsequence (relabeled) v_n which converges weakly to v_0 in $H^1(\Omega)$, and v_n converges strongly to v_0 in $L^2(\partial\Omega)$. Without loss of generality $\lambda_n \to \lambda_0 \in [0,1]$. Due to the at most linear growth condition on the boundary nonlinearity g, it follows that $\dfrac{g(x,u_n)}{||u_n||_c}$ is bounded in $L^2(\partial\Omega)$. Using the fact that $L^2(\partial\Omega)$ is a reflexive Banach space, we get that $\dfrac{g(x,u_n)}{||u_n||_c}$ converges weakly to g_0 in $L^2(\partial\Omega)$. Dividing (2.6) by $||u_n||_c$ we get that

$$\int \nabla v_n \nabla v + \int c(x) v_n v = (1-\lambda_n)\sigma \oint v_n v + \lambda_n \oint \frac{g(x,u_n)}{||u_n||_c} v \quad \text{for all } v \in H^1(\Omega). \tag{2.7}$$

Going to the limit as $n \to \infty$, we have that

$$\int \nabla v_0 \nabla v + \int c(x) v_0 v = (1-\lambda_0)\sigma \oint v_0 v + \lambda_0 \oint g_0 v \quad \text{for all } v \in H^1(\Omega), \tag{2.8}$$

Taking $v = v_0$ in (2.8) we get

$$||v_0||_c^2 = (1-\lambda_0)\sigma \oint v_0^2 + \lambda_0 \oint g_0 v_0. \tag{2.9}$$

Now, taking $v = \dfrac{u_n}{||u_n||_c}$ in (2.7), we get that $1 = ||v_n||_c^2 = (1-\lambda_n)\sigma \oint v_n^2 + \lambda_n \oint \dfrac{g(x,u_n)}{||u_n||_c} v_n$. Taking the limit as $n \to \infty$ and using (2.9) and the fact that $\dfrac{g(x,u_n)}{||u_n||_c}$ converges weakly to g_0 in $L^2(\partial\Omega)$ and v_n converges strongly to v_0 in $L^2(\partial\Omega)$, we have that

$$||v_0||_c^2 = (1-\lambda_0)\sigma \oint v_0^2 + \lambda_0 \oint g_0 v_0 = 1. \tag{2.10}$$

Now, we want to show that $v_0 = 0$; which will lead to a contradiction. From (2.7), notice that v_0 is a weak solution of the following linear equation

$$\begin{cases} -\Delta u + c(x)u = 0 & \text{a.e. in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = (1-\lambda_0)\sigma u + \lambda_0 g_0 & \text{on } \partial\Omega. \end{cases} \tag{2.11}$$

Let us mention here that Eq.(2.11) implies that $\lambda_0 \neq 0$. Otherwise, since σ is in the Steklov resolvent, we deduce that $v_0 = 0$; which contradicts the fact that $||v_0||_c^2 = 1$.

In order to bring out all the properties of the function v_0, we need to analyze a little bit more carefully the function $(1-\lambda_0)\sigma v_0(x) + \lambda_0 g_0(x)$. Let us denote by $k(x)$ the function defined by

$$k(x) = \begin{cases} (1-\lambda_0)\sigma + \lambda_0 \dfrac{g_0(x)}{v_0(x)} & \text{if } v_0(x) \neq 0, \\ 0 & \text{if } \ v_0(x) = 0. \end{cases}$$

From the definition of σ and the conditions in Theorem 1, it turns out that

$$\mu_j \le \alpha(x) \le k(x) \le \beta(x) \le \mu_{j+1} \ \text{ for } v_0(x) \neq 0. \tag{2.12}$$

Therefore, v_0 is a weak solution to the linear equation

$$\begin{cases} -\Delta u + c(x)u = 0 & \text{a.e. in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = k(x)u & \text{on } \partial\Omega, \end{cases} \tag{2.13}$$

that is;

$$\int \nabla v_0 \nabla v + \int c(x) v_0 v = \oint k(x) v_0 v \quad \text{for all } v \in H^1(\Omega). \tag{2.14}$$

We claim that this implies that either $v_0 \in E_j$ or $v_0 \in E_{j+1}$ only (see Lemma 2 below). Let us assume for the time being that this holds and finish the proof.
If $v_0 \in E_j$, then taking $v = v_0$ in (2.14) we have that $\mu_j \oint v_0^2 = ||v_0||_c^2 = \oint k(x) v_0^2$. Using (2.12), we get that $\oint (\alpha(x) - \mu_j) v_0^2 \le 0$. Since $\oint (\alpha(x) - \mu_j)\varphi^2 > 0$ for all $\varphi \in E_j \setminus \{0\}$, we conclude that $v_0 = 0$; which contradicts the fact that $||v_0||_c^2 = 1$.
Similarly, if $v_0 \in E_{j+1}$, then taking again $v = v_0$ in (2.14), we get that $\oint (\mu_{j+1} - \beta(x)) v_0^2 \le 0$. Since $\oint (\mu_{j+1} - \beta(x))\psi^2 > 0$ for all $\varphi \in E_{j+1} \setminus \{0\}$, we conclude that $v_0 = 0$; which contradicts the fact that $||v_0||_c^2 = 1$ again.
Thus, all possible solutions of the homotopy (2.2) are (uniformly) bounded in $H^1(\Omega)$ independently of $\lambda \in [0,1]$. The proof is complete.
The following lemma provide some useful information about the function v_0 that was used in the proof of the preceding lemma.

Lemma 2 *If u is a (nontrivial) weak solution of Eq.(2.13) with $\mu_j \le \alpha(x) \le k(x) \le \beta(x) \le \mu_{j+1}$, then either $u \in E_j$ or $u \in E_{j+1}$.*

Proof. Since u is (also) a weak solution, it satisfies

$$\int \nabla u \nabla v + \int c(x) u v = \oint k(x) u v \quad \text{for all } v \in H^1(\Omega). \tag{2.15}$$

Observe that $u \in [H_0^1(\Omega)]^\perp$. Hence, $u = \theta + \omega$, where $\theta \in \oplus_{l \le j} E_l$ and $\theta \in \oplus_{l \ge j+1} E_l$. We know from the properties of the Steklov eigenfunctions (see e.g. [1, 9]) that

$$||\theta||_c^2 \le \mu_j \oint \theta^2 \text{ for all } \theta \in \oplus_{l\le j} E_l \quad \text{and} \quad ||\omega||_c^2 \ge \mu_{j+1} \oint \omega^2 \text{ for all } \omega \in \oplus_{l \ge j+1} E_l. \tag{2.16}$$

Taking $v = \theta - \omega$ in (2.15), we get that

$$\int |\nabla\theta|^2 + c(x)\theta^2 - \int |\nabla\omega|^2 + c(x)\omega^2 = \oint k(x)\theta^2 - \oint k(x)\omega^2. \tag{2.17}$$

Using (2.16), we obtain that $\oint (k(x) - \mu_j)\theta^2 + \oint (\mu_{j+1} - k(x))\omega^2 \le 0$. Therefore,

$$\oint (k(x) - \mu_j)\theta^2 = 0 \quad \text{and} \quad \oint (\mu_{j+1} - k(x))\omega^2 = 0.$$

Let $S_1 := \{x \in \partial\Omega : \theta(x) \neq 0\}$ and $S_2 := \{x \in \partial\Omega : \omega(x) \neq 0\}$. It follows that

$$k(x) = \mu_j \text{ a.e. on } S_1 \quad \text{and} \quad k(x) = \mu_{j+1} \text{ a.e on } S_2. \tag{2.18}$$

If $meas(S_1 \cap S_2) > 0$, we have that $\mu_j = k(x) = \mu_{j+1}$ for a.e. $x \in S_1 \cap S_2$, which cannot happen since $\mu_j \neq \mu_{j+1}$.
Now assume that $meas(S_1 \cap S_2) = 0$; that is; $\omega(x) = 0$ a.e. on S_1 and $\theta(x) = 0$ a.e. on S_2. If $\theta \not\equiv 0$, then taking $v = \theta$ in (2.15) and using the c-orthogonality, we get that

$$\int |\nabla\theta|^2 + c(x)\theta^2 = \oint k(x)\theta^2 + \oint k(x)\omega\theta = \oint k(x)\theta^2.$$

Since $k(x) \ge \mu_j$, we have that $||\theta||_c^2 \ge \mu_j \oint \theta^2$. It follows from (2.16) that $||\theta||_c^2 = \mu_j \oint \theta^2$; which implies that $\theta \in E_j$.
Similarly, if $\omega \not\equiv 0$, then taking $v = \omega$ in (2.15) and using the c-orthogonality, we get that

$$\int |\nabla\omega|^2 + c(x)\omega^2 = \oint k(x)\omega^2 + \oint k(x)\omega\theta = \oint k(x)\omega^2.$$

Since $k(x) \le \mu_{j+1}$, we have that $||\omega||_c^2 \le \mu_{j+1} \oint \omega^2$. It follows from (2.16) that $||\omega||_c^2 = \mu_{j+1} \oint \omega^2$; which implies that $\omega \in E_{j+1}$. Thus, $u = \theta + \omega$ with $\theta \in E_j$ and $\omega \in E_{j+1}$.
Finally, we claim that the function u cannot be written in the form $u = \theta + \omega$ where $\theta \in E_j \backslash \{0\}$ and $\omega \in E_{j+1} \backslash \{0\}$. Indeed, suppose that this does not hold; that is, $u = \theta + \omega$ with $\theta \in E_j \backslash \{0\}$ and $\omega \in E_{j+1} \setminus \{0\}$. Then, by taking $v = \theta - \omega$ in (2.15), we again get (2.17). Since $\theta \in E_j$ and $\omega \in E_{j+1}$ and $\alpha(x) \le k(x) \le \beta(x)$ a.e. on $\partial\Omega$, we deduce that

$$\oint (\alpha(x) - \mu_j)\theta^2 \le 0 \quad \text{and} \quad \oint (\mu_{j+1} - \beta(x))\omega^2 \le 0;$$

which contradicts the fact that $\oint (\alpha(x) - \mu_j)\varphi^2 > 0$ for all $\varphi \in E_j \setminus \{0\}$ and $\oint (\mu_{j+1} - \beta(x))\psi^2 > 0$ for all $\psi \in E_{j+1} \setminus \{0\}$. Thus, either $u \in E_j$ or $u \in E_{j+1}$. The proof is complete.
We are now in a position to prove the main result stated above.
Proof of Theorem 1 Let $(\lambda, u) \in [0,1] \times W_p^{1-1/p}(\partial\Omega)$ be a solution to the homotopy (2.5) (equivalently (2.2)). Since $\int \nabla u \nabla v + \int c(x)uv = 0$ for all $v \in C_0^1(\Omega)$, and the trace of

$u \in W_p^{1-1/p}(\partial\Omega) \subset C(\partial\Omega)$, it follows from Theorem 13.1 in [6, pp. 199-200] (also see [3, 4]) that there is a constant $c_0 > 0$ (independent of u) such that $\sup_{\Omega} |u(x)| \leq c_0 |u|_{H^1(\Omega)}$, and so $\max_{\overline{\Omega}} |u(x)| \leq c_0 |u|_{H^1(\Omega)}$ by continuity of u on $\overline{\Omega}$. From Lemma 1 above and the (local Lipschitz) continuity of g we deduce that that $\max_{\partial\Omega} |\partial u/\partial\nu| = \max_{\partial\Omega} |(1-\lambda)\sigma u + \lambda g(\cdot, u)|$ is bounded independently of u and λ. Actually, we deduce from Theorem 2 in [7, p. 1204] that $|u|_{C^1(\overline{\Omega})}$ is bounded (independently of u and λ). Therefore, the continuity of the trace operator $C^1(\overline{\Omega}) \subset W_p^1(\Omega) \to W_p^{1-1/p}(\partial\Omega)$ and Lemma 1 herein imply that there is a constant $c_1 > 0$ (independent of u and λ) such that

$$|u|_{W_p^{1-1/p}(\partial\Omega)} < c_1, \tag{2.19}$$

for all possible solutions to the homotopy (2.5) (or equivalently (2.2)).
Now, by the homotopy invariance property of the topological degree (see e.g. [8, 12]), it follows that

$$1 = deg(I, B_{c_1}(0), 0) = deg(I - \mathcal{KN}, B_{c_1}(0), 0) \neq 0,$$

where $B_{c_1}(0) \subset W_p^{1-1/p}(\partial\Omega)$ is the ball of radius $c_1 > 0$ centered at the origin. Thus, by the existence property of the topological degree (see e.g. [8, 12]), the (nonlinear) operator $\mathcal{KN}$ has a fixed point in $W_p^{1-1/p}(\partial\Omega)$ (which is also in $W_p^2(\Omega)$ as aforementioned). The proof is complete.

Remark 1 Notice that, since g is locally Lipschitz, it follows from (2.19) and the boundary condition in the homotopy (2.2) that $|u|_{W_p^{2-1/p}(\partial\Omega)} \leq c_2$ for some constant $c_2 > 0$ independent of u and λ. Therefore, $|u|_{W_p^2(\Omega)} \leq c_3$ for some constant $c_3 > 0$.

Remark 2 The case $\mu_j = \mu_1$ more clearly illustrates the fact that the nonresonance conditions in Theorem 1 are genuinely of nonuniform type. Indeed, in this case $E_1 \setminus \{0\}$ contains only (continuous) functions which are either positive or negative on $\overline{\Omega}$. The condition that $\alpha(x) \geq \mu_1$ a.e. on $\partial\Omega$ with $\oint (\alpha(x) - \mu_1)\varphi^2 > 0$ for all $\varphi \in E_1 \setminus \{0\}$ is equivalent to saying that $\alpha(x) \geq \mu_1$ a.e. on $\partial\Omega$ with strict inequality on a subset of $\partial\Omega$ of positive measure. Thus $\alpha(x)$ need not be (uniformly) bounded away from μ_1.

Remark 3 Our main result, Theorem 1 herein, still holds true when $c \equiv 0$. (This Laplace's equation is the original linear equation which was considered by Steklov on a disk in [15].) Indeed, a modification is needed in the proof of Lemma 1 as follows. We proceed as in that proof with $||\cdot||_c$ replaced by $||\cdot||_1$ (here $||\cdot||_1$ denotes the standard $H^1(\Omega)$-norm), and $v_n = u_n/||u_n||_1$ up to the equation (2.8). Taking $v = v_0$ in (2.8) we now get

$$\int |\nabla v_0|^2 = (1-\lambda_0)\sigma \oint v_0^2 + \lambda_0 \oint g_0 v_0. \tag{2.20}$$

Now, taking $v = u_n/||u_n||_1$ in (2.7) where $||u_n||_c$ is replaced by $||u_n||_1$, we get that

$$\int |\nabla v_n|^2 = (1-\lambda_n)\sigma \oint v_n^2 + \lambda_n \oint \frac{g(x, u_n)}{||u_n||_1} v_n. \tag{2.21}$$

Taking the limit as $n \to \infty$ and using (2.20) and the fact that $\dfrac{g(x,u_n)}{||u_n||_1}$ converges weakly to g_0 in $L^2(\partial\Omega)$ and v_n converges strongly to v_0 in $L^2(\partial\Omega)$, we have that

$$\lim_{n\to\infty} \int |\nabla v_n|^2 = (1-\lambda_0)\sigma \oint v_0^2 + \lambda_0 \oint g_0 v_0 = \int |\nabla v_0|^2.$$

This implies that

$$||v_0||_1^2 = \int |\nabla v_0|^2 + \int v_0^2 = \lim_{n\to\infty} \left(\int |\nabla v_n|^2 + \int v_n^2 \right) = \lim_{n\to\infty} ||v_n||_1^2 = 1. \tag{2.22}$$

We now proceed as in the proof of Lemma 1 after Eq.(2.10) to show that $v_0 = 0$; which is a contradiction with (2.22).

The proof of Lemma 2 also needs to be modified as follows. The norm $||\cdot||_c$ is now replaced by the $H^1(\Omega)$-equivalent norm $||\cdot||$ defined by

$$||u||^2 := \int |\nabla u|^2 + \oint u^2 \quad \text{for } u \in H^1(\Omega).$$

(See e.g. [1, pp. 333–334].) By using the decomposition of $H^1(\Omega)$ given in [1, Theorem 7.3, p. 337], we now proceed with the arguments used in the proof of Lemma 2 herein to reach its conclusion.

Remark 4 Note that the case $c \equiv 0$ even more clearly illustrates the fact that the nonresonance conditions in Theorem 1 are genuinely of nonuniform type. Indeed, in this case $\mu_1 = 0$ and E_1 contains only constant functions. The condition that $\alpha(x) \geq \mu_1$ a.e. on $\partial\Omega$ with $\oint (\alpha(x) - \mu_1)\varphi^2 > 0$ for all $\varphi \in E_1 \setminus \{0\}$ is equivalent to $\alpha(x) \geq 0$ a.e. on $\partial\Omega$ with strict inequality on a subset of $\partial\Omega$ of positive measure. Thus $\alpha(x)$ need not be (uniformly) bounded away from $\mu_1 = 0$. Actually, a careful analysis of the proofs of lemmas 1 and 2 shows that, in this case, one can drop the requirement that $\alpha(x) \geq 0$ a.e. on $\partial\Omega$ and require only that $\oint \alpha(x) > 0$. Thus, a 'crossing' of the zero eigenvalue on a subset of $\partial\Omega$ of positive measure is allowed; that is, $\alpha(x)$ could be negative on a subset of $\partial\Omega$ of positive measure.

Remark 5 Our main result remains valid if one considers an equation with a more general linear part with variable coefficients; that is,

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + c(x)u = 0 & \text{a.e. in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = g(x,u) & \text{on } \partial\Omega, \end{cases} \tag{2.23}$$

where now $\partial/\partial\nu := \nu \cdot A\nabla$ is the (unit) outward conormal derivative. The matrix $A(x) := \big(a_{ij}(x)\big)$ is symmetric with $a_{ij} \in C^{0,1}(\overline{\Omega})$ such that there is a constant $\gamma > 0$ such that for all $\xi \in \mathbb{R}^n$,

$$\langle A(x)\xi, \xi\rangle \geq \gamma|\xi|^2 \quad \text{on } \overline{\Omega}.$$

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